

Physical realizability of small-world networks

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Supplementing a lattice with long-range connections effectively models small-world networks characterized by a high local and global interconnectedness observed in systems ranging from society to the brain. If the links have a wiring cost associated with their length l , the corresponding distribution $q(l)$ plays a crucial role. Uniform length distributions have received the most attention despite indications that $q(l) \sim l^{-\alpha}$ exists—e.g., for integrated circuits, the Internet, and cortical networks. While length distributions of this type were previously examined in the context of navigability, we here discuss for such systems the emergence and physical realizability of small-world topology. Our simple argument allows us to understand under which condition and at what expense a small world results.

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The explosion of research activity in the field of complex networks has led to a framework in order to describe systems in disciplines ranging from the social sciences to biology [1]. One feature shared by most real networks is the *small-world* (SW) property involving a high degree of interconnectedness at both a local and global level. That is, for every node, most nodes close to it should also be close to each other and every pair of nodes is separated, on average, by only a few links [2]. More precisely, the latter is usually expressed with an at most logarithmic increase of the mean distance as a function of the system size. Although the SW phenomenon was first introduced in a social context [3], it is also relevant for communication and technological systems such as the Internet [4] or electronic circuits [5]. Small-world properties are of great relevance for communication systems: SW networks are particularly efficient for message passing protocols that rely only on local knowledge of the network available to each node [6]. It has also been pointed out recently that SW networks could describe the architecture of neuronal networks: *in vitro* neuronal networks [7] and brain functional networks [8] as well as the cerebral cortex [9] exhibit SW features. In fact, the topology plays a crucial role in a neural network, since the high local interconnectedness gives rise to coherent oscillations while short global distances ensure a fast system response [10].

To model SW networks in Euclidean space, one starts with a regular lattice which is highly interconnected locally and then rewires every link (connecting nodes A and B) with probability p ; that is, the edge between the vertices A and B is replaced by a long-range connection (or *shortcut*) between nodes A and C , C being chosen at random [2]. Clearly, the short global distances are due to the presence of shortcuts, and as described in more detail below, it is the aim of this paper to investigate the physical realizability of a SW net-

work. In the above model, p allows one to interpolate continuously between a fully regular ($p=0$) and an entirely random ($p=1$) topology, the precise nature of this transition being discussed below. If the shortcuts are merely added (without losing local connections), no significant changes in the emergence of the SW topology result. We therefore deal with the model where rewiring is not accompanied by edge removal.

In the original formulation of the SW model, which received most of the attention [11], the length distribution of the shortcuts is uniform, since a node can choose any other node to establish a shortcut, irrespective of their Euclidean distance. Yet new interesting properties emerge if this condition is relaxed—for example, if the distribution $q(l)$ of connection lengths l decays as a power law, $q(l) \sim l^{-\alpha}$. The navigability in such a small world, for example, depends on the decay exponent α [12], and the nature of random walks and diffusion over the network is also affected [13,14]. It was even conjectured that the fundamental mechanism behind the SW phenomenon is neither disorder nor randomness, but rather the presence of multiple length scales [15] in agreement with $q(l) \sim l^{-\alpha}$. Here we establish the properties of the wiring mechanism which allows one to realize SW networks, the improved navigability being a consequence of the SW property.

Real SW networks are unlikely to be successfully modeled according to the Watts-Strogatz recipe given above: if shortcuts have to be physically realized, the cost of a long-range connection is likely to grow with its length. Since nodes connected by shortcuts can be at any Euclidean distance from each other, it turns out that the amount of resources that they have to invest in their connections grows linearly with the linear system size, and it is, *a priori*, unpredictable. This is far from optimal for systems composed of entities with limited resources (e.g. providers or neurons). Indeed, local (single-node) and global *wiring cost* considerations are likely to be key factors in the formation of real SW networks [16–21]. Regarding connection-length distributions $q(l) \sim l^{-\alpha}$, such measurements were reported for systems created through self-organization, design, and evolution—

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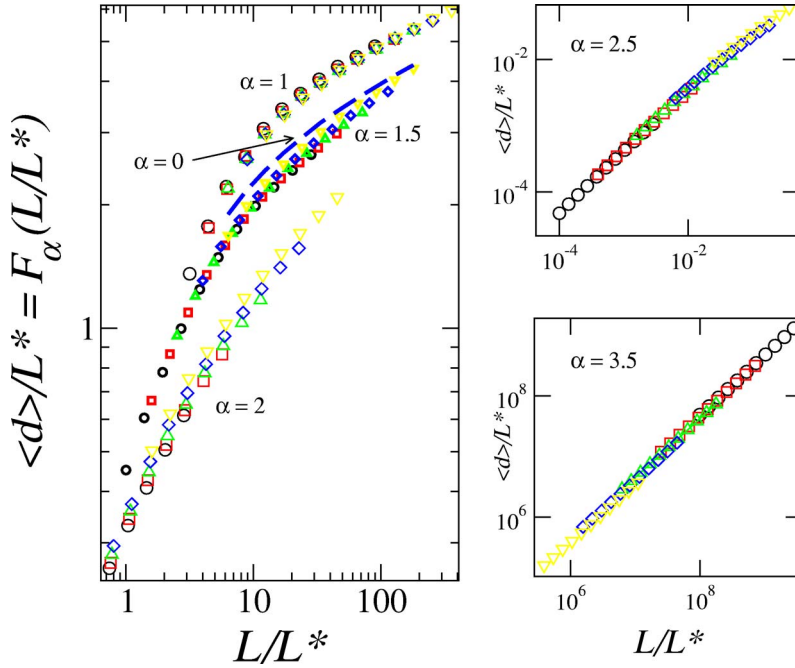


FIG. 1. (Color online) Mean distance versus linear system size, both of these quantities being rescaled by $L_\alpha^*(p)$, for $p=0.001$ (\circ), $p=0.002$ (\square), $p=0.004$ (\triangle), $p=0.008$ (\diamond), and $p=0.016$ (∇). The exponent α ranges from 0 to 3.5 as indicated. The data collapses confirm Eqs. (5) and (1) also for $\alpha > \alpha_c$ (lower right panel).

namely, for the Internet [21], integrated circuits [22], the human cortex [23], and regions of the human brain correlated at the functional level [8]. Some modeling effort taking into account the constraint of wiring minimization has been made for systems where the connection lengths are [24] or are not distributed according to a power law [25–27], and such length distributions emerge quite naturally when wiring costs along with shortest paths are minimized [28].

In this work we reanalyze the SW phenomenon from a wiring cost perspective, for networks in D dimensions, built using a power-law decaying distribution of shortcut lengths. We find, both analytically and numerically, that $\alpha < D + 1$ is the condition for the emergence of SW behavior. We also found that the local interconnectedness increases with α and, given a fixed total wiring cost, networks with larger values of α are smaller worlds.

Given a D -dimensional lattice of linear size L , consisting of $N=L^D$ sites, subject to periodic boundary conditions, it shall be supplemented with pN additional connections whose lengths are distributed according to $q(l) \sim l^{-\alpha}$ as follows: for every link to be added, we first choose its length according to the (one-dimensional) distribution $q(l)$ and then put it on the lattice by randomly choosing one end point and the other at the drawn distance l , such that no pair of sites is connected by more than one additional connection.

Clearly, a certain amount of real shortcuts—i.e., long additional links—is required for SW topology to emerge [2,31]. It can thus be anticipated that the exponent α has to be smaller than a critical value α_c . Before we give the argument allowing one to derive α_c , let us recall that SW topology is characterized by the following behavior of the mean distance:

$$\langle d \rangle = L^* \mathcal{F}_\alpha \left(\frac{L}{L^*} \right), \quad (1)$$

the scaling function obeying [31,32]

$$\mathcal{F}_\alpha(x) \sim \begin{cases} x & \text{if } x \ll 1, \\ \ln(x) & \text{if } x \gg 1. \end{cases} \quad (2)$$

In other words, SW topology corresponds to a logarithmic increase of the mean distance with the system size ($L \gg L^*$) whereas in a *large world* (LW)—i.e., if $L \ll L^*$ — $\langle d \rangle \sim L$. For $\alpha=0$, the critical length scale in Eq. (1) is given by $L^*(p) \sim p^{-1/D}$ [32,33]. If α is positive, we shall derive L^* (as well as α_c) through the following indirect argument: We look at the probability that an arbitrarily chosen additional link is a real shortcut, that is, that it spans the lattice,

$$P_c(L) = \int_{(1-c)L/2}^{L/2} q(l) dl, \quad (3)$$

c being small but finite, and require (our ansatz) the expected number of such connections to be of the order of 1 [31]:

$$P_c(L)[p^*(L)L^D] \approx 1. \quad (4)$$

Here $p^*(L)L^D$ is the desired number of additional links, implying the emergence of SW topology for $p \gg p^*(L)$. After evaluating the scaling of Eq. (3), Eq. (4) reads

$$p^*(L) \sim \begin{cases} L^{-D} & \text{if } \alpha < 1, \\ \ln(L)/L^D & \text{if } \alpha = 1, \\ L^{\alpha-D-1} & \text{if } \alpha > 1. \end{cases} \quad (5)$$

Equation (5) implies $L^*(p) \sim p^{-1/D}$ for $\alpha < 1$; i.e., the behavior of L^* in this α range is the same as that for $\alpha=0$. In the case $\alpha > 1$, we have $L^*(p) \sim p^{1/(\alpha-D-1)}$, thus becoming infinite at

$$\alpha_c = D + 1. \quad (6)$$

We therefore have two possible regimes for $\alpha < \alpha_c$ while LW behavior prevails for $\alpha \geq \alpha_c$. Figure 1 shows the rescaled mean distances as a function of the rescaled linear system

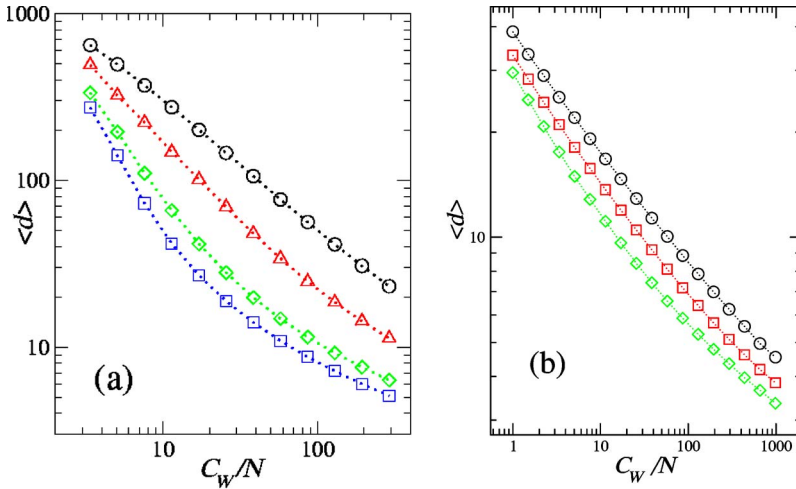


FIG. 2. (Color online) (a) Mean distance as a function of the total wiring costs (divided by the number of sites) for one-dimensional topologies ($N=10^4$). The curves (\circ , $\alpha=0$; \triangle , $\alpha=1$; \diamond , $\alpha=1.5$; and \square , $\alpha=1.75$) show that the mean distance decreases with α for a fixed value of C_w/N . (b) Analogous results for $D=2$ [$N=500 \times 500$ and $\alpha=0$ (\circ), $\alpha=1$ (\square) and $\alpha=2$ (\diamond)]. All the points shown here result from averaging over 100 realizations of networks.

size for different values of α and $p=0.001, 0.002, \dots, 0.016$ in each set for the case $D=2$. The observed data collapses for all chosen values of α confirm Eq. (5) obtained by our simple argument as well as Eq. (1). We numerically verified Eq. (2), especially in the limit $L/L^* \ll 1$, the logarithmic tail of \mathcal{F}_α further being exhibited best for small α .¹

As outlined above, a SW network is also characterized by a high *local* interconnectedness. This topological property can, for example, be measured by the clustering coefficient C which is the probability that two nodes are connected, given that they share a nearest neighbor. In contrast to the Watts-Strogatz model, our initial lattices are characterized by $C=0$, but by increasing the exponent of the link-length distribution, the degree of clustering becomes orders of magnitudes larger than for random networks with the same number of nodes and links.

Let us now examine the wiring costs, which were our prime motivation to look at SW networks with power-law decaying link-length distributions and an important ingredient for real SW networks. The moments $\langle l \rangle$ and $\langle l^2 \rangle$ play a crucial role as far as these costs are concerned. Indeed, finite $\langle l \rangle$ and $\langle l^2 \rangle$ would allow for predictable costs for each node and consequently for a better design of the network constituents. The total wiring cost $C_w = pL^D \langle l \rangle$ is also an important quantity, its minimization governing, for example, the evolution of cortical networks [17]. We find for the first two moments the scaling relations summarized in Table I, the expressions for integer α being modified by logarithmic corrections. In two dimensions, SW topology can be realized even if $\langle l \rangle = \text{const}$ (that is, for $2 < \alpha < 3 = \alpha_c$) whereas this is not the case in one dimension where $\langle l \rangle$ becomes finite in the $L \rightarrow \infty$ limit only above $\alpha_c = 2$. Moreover, if $D=3$, it is even possible to have $\langle l \rangle = O(1) = \langle l^2 \rangle$ while still being in the SW regime for $3 < \alpha < 4 = \alpha_c$. An appropriate choice of the parameters D and α is thus the key to modeling networks which are both efficient (SW topology) and economical (low wiring costs).

¹For general α , the second line of Eq. (2) may also read $[\ln(x)]^{s(\alpha)}$, $s(\alpha) > 0$.

It is furthermore interesting to have a closer look at the relationship between the wiring costs and the topology. As α varies, one can ask what mean distance results given a total amount of wiring length for the additional connections (i.e., the total cost). Figure 2(a) reports these dependences for $\alpha=0, 1, 1.5$, and 1.75 (going from the uppermost to the lowest set) for one-dimensional topologies of 10^4 sites. The largest value of $\langle d \rangle$ (the leftmost circle) corresponds to the length scale $L^* < 10^3 \ll 10^4 = L$; thus, all the points in the figure represent the system in the SW regime. It can clearly be seen that the mean distance decreases with α at fixed wiring costs C_w/N —i.e., the larger α , the smaller the world. This behavior is qualitatively recovered when expressing Eq. (1) in terms of $x = C_w/N = p\langle l \rangle$. We made similar observations in two dimensions [see Fig. 2(b)].

Let us now point out the generality of our argument for the realizability of SW networks in Euclidean space. In fact, it also applies to a version of the SW model where the links are added in a different way: at every site, a link is added with probability p —the other end point being chosen according to the (D -dimensional) distribution $q(l)$ [29]. This procedure differs from the previous one in that the site from which the new link will emanate “sees” the dimensionality of the lattice, giving rise to a different normalization of $q(l)$ with respect to the version treated above. Furthermore, the just described mechanism is equivalent to adding a link between any pair of sites \mathbf{x} and \mathbf{y} with a probability proportional to $|\mathbf{x} - \mathbf{y}|^{-\alpha}$ [30].

For this new construction procedure, the length distribution reads

TABLE I. Behavior of the moments of the shortcut-length distribution as a function of the linear system size L (for the “adding” procedure 1).

	$0 \leq \alpha < 1$	$1 < \alpha < 2$	$2 < \alpha < 3$	$\alpha > 3$
$\langle l \rangle$	L	$L^{2-\alpha}$	const	const
$\langle l^2 \rangle$	L^2	$L^{3-\alpha}$	$L^{3-\alpha}$	const

$$q(l) = \frac{l^{D-1-\alpha}}{\int_2^{L/2} \tilde{l}^{D-1-\alpha} d\tilde{l}},$$

where the factor l^{D-1} explicitly accounts for the normalization in D -dimensional space. With Eqs. (3) and (4), which do not depend on the details of the “adding” mechanism, we obtain, for the critical probability,

$$p^*(L) \sim \begin{cases} L^{-D} & \text{if } \alpha < D, \\ \ln(L)/L^D & \text{if } \alpha = D, \\ L^{\alpha-2D} & \text{if } \alpha > D. \end{cases}$$

Conversely, this implies $L^* \sim p^{1/(\alpha-2D)}$ for $\alpha > D$ and hence the existence of a SW regime as long as

$$\alpha_c < 2D, \quad (7)$$

in analogy with the previous reasoning. Inequality (7) had already been derived [29,30], but in a less intuitive framework.

In summary, we have given a simple argument leading to the precise conditions under which small-world topology emerges and examined the physical realizability of such networks. Due to the generality of our argument, it is also applicable to other small-world models. We further showed that small-world networks can be constructed in a very economical way if the parameters D and α are chosen appropriately (although of course in real systems D is seldom a tunable parameter). As length distributions of the type investigated here have been observed in a number of real-world networks, such as integrated circuits, the Internet, or the human cortex, we believe this work to have intriguing implications in their modeling.

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